

On Asymptotics Related to Classical and Bayesian Inference in Stochastic Differential Equations with Time-Varying Covariates

Trisha Maitra and Sourabh Bhattacharya*

Abstract

Delattre *et al.* (2013) considered n independent stochastic differential equations (*SDE*'s), where in each case the drift term is modeled by a random effect times a known function free of parameters. The distribution of the random effects are assumed to depend upon unknown parameters which are to be learned about. Assuming the independent and identical (*iid*) situation the authors provide independent proofs of consistency and asymptotic normality of the maximum likelihood estimators (*MLE*'s) of the hyper-parameters of their random effects parameters.

In this article, we generalize the random effect term by incorporating time-dependent covariates and consider both fixed and random effects set-ups. We also allow the functional part associated with the drift function to depend upon unknown parameters. In this general set-up of *SDE* system we establish consistency and asymptotic normality of the *MLE* through verification of the regularity conditions required by existing relevant theorems. Besides, we consider the Bayesian approach to learning about the population parameters, and prove consistency and asymptotic normality of the corresponding posterior distribution.

Keywords: *Asymptotic normality; Maximum likelihood estimator; Posterior consistency; Random effects; Stochastic differential equations; Time-varying covariates.*

1 Introduction

Systems of stochastic differential equations (*SDE*'s) are appropriate for modeling situations where “within” subject variability is caused by some random component varying continuously in time. When suitable time-varying covariates are available, it is then appropriate to incorporate such information in the *SDE* system. Some examples of statistical applications of *SDE*-based models with time-dependent covariates are Oravec *et al.* (2011), Overgaard *et al.* (2005), Leander *et al.* (2015).

However, asymptotic inference in systems of *SDE* based models consisting of time-varying covariates seem to be rare in the statistical literature, in spite of their importance. For asymptotic inference regarding the parameters, however, so far only random effects *SDE* systems without covariates have been considered (Delattre *et al.* (2013), Maitra and Bhattacharya (2016c), Maitra and Bhattacharya (2015)). Such models are of the following form:

$$dX_i(t) = b(X_i(t), \phi_i)dt + \sigma(X_i(t))dW_i(t), \quad \text{with } X_i(0) = x^i, \quad i = 1, \dots, n. \quad (1.1)$$

where, for $i = 1, \dots, n$, $X_i(0) = x^i$ is the initial value of the stochastic process $X_i(t)$, which is assumed to be continuously observed on the time interval $[0, T_i]$; $T_i > 0$ assumed to be known. The function $b(x, \phi)$, which is the drift function, is a known, real-valued function on $\mathbb{R} \times \mathbb{R}^d$ (\mathbb{R} is the real line and d is the dimension), and the function $\sigma : \mathbb{R} \mapsto \mathbb{R}$ is the known diffusion coefficient. The *SDE*'s given by (1.1) are driven by independent standard Wiener processes $\{W_i(\cdot); i = 1, \dots, n\}$, and $\{\phi_i; i = 1, \dots, n\}$, which are to be interpreted as the random effect parameters associated with the n individuals, which are assumed by Delattre *et al.* (2013) to be independent of the Brownian motions and independently and identically distributed (*iid*) random variables with some common distribution.

For the sake of convenience Delattre *et al.* (2013) (see also Maitra and Bhattacharya (2016c) and Maitra and Bhattacharya (2015)) assume $b(x, \phi_i) = \phi_i b(x)$. Thus, the random effect is a multiplicative factor of the drift function; also, the function $b(x)$ is assumed to be independent of parameters. In this article, we generalize the multiplicative factor to include time-dependent covariates; we also allow $b(x)$ to depend upon unknown parameters.

*Trisha Maitra is a PhD student and Sourabh Bhattacharya is an Associate Professor in Interdisciplinary Statistical Research Unit, Indian Statistical Institute, 203, B. T. Road, Kolkata 700108. Corresponding e-mail: sourabh@isical.ac.in.

Notably, such model extension has already been provided in Maitra and Bhattacharya (2016a) and Maitra and Bhattacharya (2016b), but their goal was to develop asymptotic theory of Bayes factors for comparing systems of SDE 's, with or without time-varying covariates, emphasizing, when time-varying covariates are present, simultaneous asymptotic selection of covariates and part of the drift function free of covariates, using Bayes factors.

In this work, we deal with parametric asymptotic inference, both classical and Bayesian, in context of our extended system of SDE 's. We consider, separately, fixed effects as well as random effects. The fixed effects set-up ensues when coefficients associated with the covariates are the same for all the subjects. On the other hand, in the random effects set-up, the subject-wise coefficients are assumed to be a random sample from some distribution with unknown parameters.

It is also important to distinguish between the *iid* situation and the independent but non-identical case (we refer to the latter as non-*iid*) that we consider. The *iid* set-up is concerned with the case where the initial values x^i and time limit T_i are the same for all i , and the coefficients associated with the covariates are zero, that is, there are no covariates suitable for the SDE -based system. This set-up, however, does not reduce to the *iid* set-up considered in Delattre *et al.* (2013), Maitra and Bhattacharya (2016c) and Maitra and Bhattacharya (2015) because in the latter works $b(x)$ was assumed to be free of parameters, while in this work we allow this function to be dependent on unknown parameters. The non-*iid* set-up assumes either or both of the following: presence of appropriate covariates and that x^i and T_i are not the same for all the subjects.

In the classical paradigm, we investigate consistency and asymptotic normality of the maximum likelihood estimator (MLE) of the unknown parameters which we denote by θ , and in the Bayesian framework we study consistency and asymptotic normality of the Bayesian posterior distribution of θ . In other words, we consider prior distributions $\pi(\theta)$ of θ and study the properties of the corresponding posterior

$$\pi_n(\theta|X_1, \dots, X_n) = \frac{\pi(\theta) \prod_{i=1}^n f_i(X_i|\theta)}{\int_{\psi \in \Theta} \pi(\psi) \prod_{i=1}^n f_i(X_i|\psi) d\psi} \quad (1.2)$$

as the sample size n tends to infinity. Here $f_i(\cdot|\theta)$ is the density corresponding to the i -th individual and Θ is the parameter space.

In what follows, after introducing our model and concerned likelihood in Section 2, we investigate asymptotic properties of MLE in the *iid* and non-*iid* contexts in Sections 3 and 4 respectively. Then, in Sections 5 and 6 we investigate asymptotic properties of the posterior in the *iid* and non-*iid* cases, respectively. In Section 7 we consider the random effects set-up and provide necessary discussion to point towards validity of the corresponding asymptotic results. We summarize our contribution and provide further discussion in Section 8.

Notationally, “ $\xrightarrow{a.s.}$ ”, “ \xrightarrow{P} ” and “ $\xrightarrow{\mathcal{L}}$ ” denote convergence “almost surely”, “in probability” and “in distribution”, respectively.

2 The concerned SDE set-up

We consider the following two systems of SDE models for $i = 1, 2, \dots, n$:

$$dX_i(t) = \phi_{i,\xi}(t)b_\beta(X_i(t))dt + \sigma(X_i(t))dW_i(t) \quad (2.1)$$

where, $X_i(0) = x^i$ is the initial value of the stochastic process $X_i(t)$, which is assumed to be continuously observed on the time interval $[0, T_i]$; $T_i > 0$ for all i and assumed to be known.

2.1 Incorporation of time-varying covariates

We assume that $\phi_{i,\xi}(t)$ has the following form:

$$\phi_{i,\xi}(t) = \phi_{i,\xi}(z_i(t)) = \xi_0 + \xi_1 g_1(z_{i1}(t)) + \xi_2 g_2(z_{i2}(t)) + \dots + \xi_p g_p(z_{ip}(t)), \quad (2.2)$$

where $\xi = (\xi_0, \xi_1, \dots, \xi_p)$ is a set of real constants and $z_i(t) = (z_{i1}(t), z_{i2}(t), \dots, z_{ip}(t))$ is the set of available covariate information corresponding to the i -th individual, depending upon time t . We assume that $z_i(t)$ is continuous in t , $z_{il}(t) \in \mathbf{Z}_l$ where \mathbf{Z}_l is compact and $g_l : \mathbf{Z}_l \rightarrow \mathbb{R}$ is continuous, for $l = 1, \dots, p$. Let $\mathbf{Z} = \mathbf{Z}_1 \times \dots \times \mathbf{Z}_p$. We let $\mathfrak{Z} = \{z(t) \in \mathbf{Z} : t \in [0, \infty) \text{ such that } z(t) \text{ is continuous in } t\}$. Hence, $z_i \in \mathfrak{Z}$ for all i . The function b_β is multiplicative part of the drift functions free of the covariates. Note that ξ consists of $p + 1$ number of parameters. Assuming that $\beta \in \mathbb{R}^q$, where $q \geq 1$, it follows that our parameter set $\theta = (\beta, \xi)$ belongs to the $(p + q + 1)$ -dimensional real space \mathbb{R}^{p+q+1} . The true parameter set is denoted by θ_0 .

2.2 Likelihood

We first define the following quantities:

$$U_{i,\theta} = \int_0^{T_i} \frac{\phi_{i,\xi}(s) b_\beta(X_i(s))}{\sigma^2(X_i(s))} dX_i(s), \quad V_{i,\theta} = \int_0^{T_i} \frac{\phi_{i,\xi}^2(s) b_\beta^2(X_i(s))}{\sigma^2(X_i(s))} ds \quad (2.3)$$

for $i = 1, \dots, n$.

Let \mathcal{C}_{T_i} denote the space of real continuous functions $(x(t), t \in [0, T_i])$ defined on $[0, T_i]$, endowed with the σ -field \mathcal{C}_{T_i} associated with the topology of uniform convergence on $[0, T_i]$. We consider the distribution P^{x_i, T_i, z_i} on $(\mathcal{C}_{T_i}, \mathcal{C}_{T_i})$ of $(X_i(t), t \in [0, T_i])$ given by (2.1). We choose the dominating measure P_i as the distribution of (2.1) with null drift. So,

$$\frac{dP^{x_i, T_i, z_i}}{dP_i} = f_i(X_i | \theta) = \exp \left(U_{i,\theta} - \frac{V_{i,\theta}}{2} \right). \quad (2.4)$$

3 Consistency and asymptotic normality of MLE in the iid set-up

In the *iid* set up we have $x_i = x$ and $T_i = T$ for all $i = 1, \dots, n$. Moreover, the covariates are absent, that is, $\xi_i = 0$ for $i = 1, \dots, p$. Hence, the resulting parameter set in this case is $\theta = (\beta, \xi_0)$.

3.1 Strong consistency of MLE

Consistency of the MLE under the *iid* set-up can be verified by validating the regularity conditions of the following theorem (Theorems 7.49 and 7.54 of Schervish (1995)); for our purpose we present the version for compact parameter space.

Theorem 1 (Schervish (1995)) *Let $\{X_n\}_{n=1}^\infty$ be conditionally iid given θ with density $f_1(x|\theta)$ with respect to a measure ν on a space $(\mathcal{X}^1, \mathcal{B}^1)$. Fix $\theta_0 \in \Theta$, and define, for each $\mathbf{M} \subseteq \Theta$ and $x \in \mathcal{X}^1$,*

$$Z(\mathbf{M}, x) = \inf_{\psi \in \mathbf{M}} \log \frac{f_1(x|\theta_0)}{f_1(x|\psi)}.$$

Assume that for each $\theta \neq \theta_0$, there is an open set \mathbf{N}_θ such that $\theta \in \mathbf{N}_\theta$ and that $E_{\theta_0} Z(\mathbf{N}_\theta, X_i) > -\infty$. Also assume that $f_1(x|\cdot)$ is continuous at θ for every θ , a.s. $[P_{\theta_0}]$. Then, if $\hat{\theta}_n$ is the MLE of θ corresponding to n observations, it holds that $\lim_{n \rightarrow \infty} \hat{\theta}_n = \theta_0$, a.s. $[P_{\theta_0}]$.

3.1.1 Assumptions

We assume the following conditions:

(H1) The parameter space $\Theta = \mathfrak{B} \times \Gamma$ such that \mathfrak{B} and Γ are compact.

(H2) $b_\beta(\cdot)$ and $\sigma(\cdot)$ are C^1 (differentiable with continuous first derivative) on \mathbb{R} and satisfy $b_\beta^2(x) \leq K_1(1 + x^2 + \|\beta\|^2)$ and $\sigma^2(x) \leq K_2(1 + x^2)$ for all $x \in \mathbb{R}$, for some $K_1, K_2 > 0$. Now, due to

(H1) the latter boils down to assuming $b_{\beta}^2(x) \leq K(1+x^2)$ and $\sigma^2(x) \leq K(1+x^2)$ for all $x \in \mathbb{R}$, for some $K > 0$.

We further assume:

(H3) For every x , let b_{β} be continuous in $\beta = (\beta_1, \dots, \beta_q)$ and moreover, for $j = 1, \dots, q$,

$$\sup_{\beta \in \mathfrak{B}} \frac{\left| \frac{\partial b_{\beta}(x)}{\partial \beta_j} \right|}{\sigma^2(x)} \leq c(1 + |x|^{\gamma}),$$

for some $c > 0$ and $\gamma \geq 0$.

(H4)

$$\frac{b_{\beta}^2(x)}{\sigma^2(x)} \leq K_{\beta} (1 + x^2 + \|\beta\|^2), \quad (3.1)$$

where K_{β} is continuous in β .

3.1.2 Verification of strong consistency of MLE in our SDE set-up

To verify the conditions of Theorem 1 in our case, note that assumptions (H1) – (H4) clearly imply continuity of the density $f_1(x|\theta)$ in the same way as the proof of the Proposition 2 of Delattre *et al.* (2013). It follows that U_{θ} and V_{θ} are continuous in θ , the property that we use in our proceedings below.

Now consider,

$$\begin{aligned} Z(\mathbf{N}_{\theta}, X) &= \inf_{\theta_1 \in \mathbf{N}_{\theta}} \log \frac{f_1(X|\theta_0)}{f_1(X|\theta_1)} \\ &= \left(U_{\theta_0} - \frac{V_{\theta_0}}{2} \right) - \inf_{\theta_1 \in \mathbf{N}_{\theta}} \left(U_{\theta_1} - \frac{V_{\theta_1}}{2} \right) \\ &\geq \left(U_{\theta_0} - \frac{V_{\theta_0}}{2} \right) - \inf_{\theta_1 \in \bar{\mathbf{N}}_{\theta}} \left(U_{\theta_1} - \frac{V_{\theta_1}}{2} \right) \\ &= \left(U_{\theta_0} - \frac{V_{\theta_0}}{2} \right) - \left(U_{\theta_1^*(X)} - \frac{V_{\theta_1^*(X)}}{2} \right) \end{aligned} \quad (3.2)$$

where \mathbf{N}_{θ} is an appropriate open subset of the relevant compact parameter space, and $\bar{\mathbf{N}}_{\theta}$ is a closed subset of \mathbf{N}_{θ} . The infimum of $\left(U_{\theta_1} - \frac{V_{\theta_1}}{2} \right)$ is attained at $\theta_1^* = \theta_1^*(X) \in \bar{\mathbf{N}}_{\theta}$ due to continuity of U_{θ} and V_{θ} in θ .

Let $E_{\theta_0}(V_{\theta_1}) = \check{V}_{\theta_1}$ and $E_{\theta_0}(U_{\theta_1}) = \check{U}_{\theta_1}$. From Theorem 5 of Maitra and Bhattacharya (2016c) it

follows that the above expectations are continuous in θ_1 . Using this we obtain

$$\begin{aligned}
E_{\theta_0} \left(U_{\theta_1^*(X)} - \frac{V_{\theta_1^*(X)}}{2} \right) &= E_{\theta_1^*(X)|\theta_0} E_{X|\theta_1^*(X)=\varphi_1, \theta_0} \left(U_{\theta_1^*(X)=\varphi_1} - \frac{V_{\theta_1^*(X)=\varphi_1}}{2} \right) \\
&= E_{\theta_1^*(X)|\theta_0} \left(\check{U}_{\varphi_1} - \frac{\check{V}_{\varphi_1}}{2} \right) \\
&\leq E_{\theta_1^*(X)|\theta_0} \left[\sup_{\varphi_1 \in \bar{\mathbf{N}}_{\theta}} \left(\check{U}_{\varphi_1} - \frac{\check{V}_{\varphi_1}}{2} \right) \right] \\
&= E_{\theta_1^*(X)|\theta_0} \left(\check{U}_{\varphi_1^*} - \frac{\check{V}_{\varphi_1^*}}{2} \right) \\
&= \left(\check{U}_{\varphi_1^*} - \frac{\check{V}_{\varphi_1^*}}{2} \right), \tag{3.3}
\end{aligned}$$

where $\varphi_1^* \in \bar{\mathbf{N}}_{\theta}$ is where the supremum of $\left(\check{U}_{\varphi_1} - \frac{\check{V}_{\varphi_1}}{2} \right)$ is achieved. Since φ_1^* is independent of X , the last step (3.3) follows.

Noting that $E_{\theta_0} \left(U_{\theta_0} - \frac{V_{\theta_0}}{2} \right)$ and $\left(\check{U}_{\varphi_1^*} - \frac{\check{V}_{\varphi_1^*}}{2} \right)$ are finite due to Lemma 1 of Maitra and Bhattacharya (2016a), it follows that $E_{\theta_0} Z(\mathbf{N}_{\theta}, X) > -\infty$. Hence, $\hat{\theta}_n \xrightarrow{a.s.} \theta_0 [P_{\theta_0}]$. We summarize the result in the form of the following theorem:

Theorem 2 Assume the iid setup and conditions (H1) – (H4). Then the MLE is strongly consistent in the sense that $\hat{\theta}_n \xrightarrow{a.s.} \theta_0 [P_{\theta_0}]$.

3.2 Asymptotic normality of MLE

To verify asymptotic normality of MLE we invoke the following theorem provided in Schervish (1995) (Theorem 7.63):

Theorem 3 (Schervish (1995)) Let Θ be a subset of \mathbb{R}^{p+q+1} , and let $\{X_n\}_{n=1}^{\infty}$ be conditionally iid given θ each with density $f_1(\cdot|\theta)$. Let $\hat{\theta}_n$ be an MLE. Assume that $\hat{\theta}_n \xrightarrow{P} \theta$ under P_{θ} for all θ . Assume that $f_1(x|\theta)$ has continuous second partial derivatives with respect to θ and that differentiation can be passed under the integral sign. Assume that there exists $H_r(x, \theta)$ such that, for each $\theta_0 \in \text{int}(\Theta)$ and each k, j ,

$$\sup_{\|\theta - \theta_0\| \leq r} \left| \frac{\partial^2}{\partial \theta_k \partial \theta_j} \log f_1(x|\theta_0) - \frac{\partial^2}{\partial \theta_k \partial \theta_j} \log f_1(x|\theta) \right| \leq H_r(x, \theta_0), \tag{3.4}$$

with

$$\lim_{r \rightarrow 0} E_{\theta_0} H_r(X, \theta_0) = 0. \tag{3.5}$$

Assume that the Fisher information matrix $\mathcal{I}(\theta)$ is finite and non-singular. Then, under P_{θ_0} ,

$$\sqrt{n} (\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} N(\mathbf{0}, \mathcal{I}^{-1}(\theta_0)). \tag{3.6}$$

3.2.1 Assumptions

Along with the assumptions (H1) – (H4), we further assume the following:

(H5) The true value $\theta_0 \in \text{int}(\Theta)$.

(H6) The Fisher's information matrix $\mathcal{I}(\theta)$ is finite and non-singular, for all $\theta \in \Theta$.

(H7) Letting $b'_\beta(x) = \frac{\partial}{\partial \beta_k} b_\beta(x)$ for $k = 1, \dots, q$; $b''_\beta(x) = \frac{\partial^2}{\partial \beta_k \partial \beta_l} b_\beta(x)$ for $k, l = 1, \dots, q$, and $b'''_\beta(x) = \frac{\partial^3}{\partial \beta_k \partial \beta_l \partial \beta_m} b_\beta(x)$ for $k, l, m = 1, \dots, q$, there exist constants $0 < c < \infty, 0 < \gamma_1, \gamma_2, \gamma_3, \gamma_4 \leq 1$ such that for each combination of $k, l, m = 1, \dots, q$, for any $\beta_1, \beta_2 \in \mathbb{R}^q$, for all $x \in \mathbb{R}$,

$$|b_{\beta_1}(x) - b_{\beta_2}(x)| \leq c \|\beta_1 - \beta_2\|^{\gamma_1};$$

$$|b'_{\beta_1}(x) - b'_{\beta_2}(x)| \leq c \|\beta_1 - \beta_2\|^{\gamma_2};$$

$$|b''_{\beta_1}(x) - b''_{\beta_2}(x)| \leq c \|\beta_1 - \beta_2\|^{\gamma_3};$$

$$|b'''_{\beta_1}(x) - b'''_{\beta_2}(x)| \leq c \|\beta_1 - \beta_2\|^{\gamma_4}.$$

3.2.2 Verification of the above regularity conditions for asymptotic normality in our SDE set-up

In Section 3.1.2 almost sure consistency of the $MLE \hat{\theta}_n$ has been established. Hence, $\hat{\theta}_n \xrightarrow{P} \theta$ under P_θ for all θ . With assumptions (H1)–(H4), (H7), Theorem B.4 of Rao (2013) and the dominated convergence theorem, interchangeability of differentiation and integration in case of stochastic integration and usual integration respectively can be assured, from which it can be easily deduced that differentiation can be passed under the integral sign, as required by Theorem 3. With the same arguments, it follows that in our case $\frac{\partial^2}{\partial \theta_k \partial \theta_j} \log f_1(x|\theta)$ is differentiable in $\theta = (\beta, \xi_0)$, and the derivative has finite expectation due to compactness of the parameter space and (H7). Hence, (3.4) and (3.5) clearly hold.

In other words, asymptotic normality of the MLE , of the form (3.6), holds in our case. Formally,

Theorem 4 Assume the iid setup and conditions (H1) – (H7). Then the MLE is asymptotically normally distributed as (3.6).

4 Consistency and asymptotic normality of MLE in the non-iid set-up

We now consider the case where the processes $X_i(\cdot)$; $i = 1, \dots, n$, are independently, but not identically distributed. In this case, $\xi = (\xi_0, \xi_1, \dots, \xi_p)$ where at least one of the coefficients ξ_1, \dots, ξ_p is non-zero, guaranteeing presence of at least one time-varying covariate. Hence, in this set-up $\theta = (\beta, \xi)$.

Moreover, following Maitra and Bhattacharya (2016c), Maitra and Bhattacharya (2015) we allow the initial values x^i and the time limits T_i to be different for $i = 1, \dots, n$, but assume that the sequences $\{T_1, T_2, \dots\}$ and $\{x^1, x^2, \dots\}$ are sequences entirely contained in compact sets \mathfrak{T} and \mathfrak{X} , respectively. Compactness ensures that there exist convergent subsequences with limits in \mathfrak{T} and \mathfrak{X} ; for notational convenience, we continue to denote the convergent subsequences as $\{T_1, T_2, \dots\}$ and $\{x^1, x^2, \dots\}$. Thus, let the limits be $T^\infty \in \mathfrak{T}$ and $x^\infty \in \mathfrak{X}$.

Henceforth, we denote the process associated with the initial value x and time point t as $X(t, x)$ and so for $x \in \mathfrak{X}$ and $T \in \mathfrak{T}$, we let

$$U_\theta(x, T, z) = \int_0^T \frac{\phi_\xi b_\beta(X(s, x))}{\sigma^2(X(s, x))} dX(s, x); \quad (4.1)$$

$$V_\theta(x, T, z) = \int_0^T \frac{\phi_\xi b_\beta^2(X(s, x))}{\sigma^2(X(s, x))} ds. \quad (4.2)$$

Clearly, $U_\theta(x^i, T_i, z_i) = U_{i,\theta}$ and $V_\theta(x^i, T_i, z_i) = V_{i,\theta}$, where $U_{i,\theta}$ and $V_{i,\theta}$ are given by (2.3). In this non-iid set-up we assume, following Maitra and Bhattacharya (2016a), that

(H8) For $l = 1, \dots, p$, and for $t \in [0, T_i]$,

$$\frac{1}{n} \sum_{i=1}^n g_l(z_{il}(t)) \rightarrow c_l(t); \quad (4.3)$$

and, for $l, m = 1, \dots, p; t \in [0, T_i]$,

$$\frac{1}{n} \sum_{i=1}^n g_l(z_{il}(t)) g_m(z_{im}(t)) \rightarrow c_l(t) c_m(t), \quad (4.4)$$

as $n \rightarrow \infty$, where $c_l(t)$ are real constants.

For $x = x^k$, $T = T_k$ and $\mathbf{z} = \mathbf{z}_k$, we denote the Kullback-Leibler distance and the Fisher's information as $\mathcal{K}_k(\boldsymbol{\theta}_0, \boldsymbol{\theta})$ ($\mathcal{K}_k(\boldsymbol{\theta}, \boldsymbol{\theta}_0)$) and $\mathcal{I}_k(\boldsymbol{\theta})$, respectively. Then the following results hold in the same way as Lemma 11 of Maitra and Bhattacharya (2016a).

Lemma 5 Assume the non-iid set-up, (H1) – (H4) and (H8). Then for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$,

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \mathcal{K}_k(\boldsymbol{\theta}_0, \boldsymbol{\theta})}{n} = \mathcal{K}(\boldsymbol{\theta}_0, \boldsymbol{\theta}); \quad (4.5)$$

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \mathcal{K}_k(\boldsymbol{\theta}, \boldsymbol{\theta}_0)}{n} = \mathcal{K}(\boldsymbol{\theta}, \boldsymbol{\theta}_0); \quad (4.6)$$

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \mathcal{I}_k(\boldsymbol{\theta})}{n} = \mathcal{I}(\boldsymbol{\theta}), \quad (4.7)$$

where the limits $\mathcal{K}(\boldsymbol{\theta}_0, \boldsymbol{\theta})$, $\mathcal{K}(\boldsymbol{\theta}, \boldsymbol{\theta}_0)$ and $\mathcal{I}(\boldsymbol{\theta})$ are well-defined Kullback-Leibler divergences and Fisher's information, respectively.

Lemma 5 will be useful in our asymptotic investigation in the non-iid set-up. In this set-up, we first investigate consistency and asymptotic normality of *MLE* using the results of Hoadley (1971).

4.1 Consistency and asymptotic normality of *MLE* in the non-iid set-up

Following Hoadley (1971) we define the following:

$$\begin{aligned} R_i(\boldsymbol{\theta}) &= \log \frac{f_i(X_i|\boldsymbol{\theta})}{f_i(X_i|\boldsymbol{\theta}_0)} \quad \text{if } f_i(X_i|\boldsymbol{\theta}_0) > 0 \\ &= 0 \quad \text{otherwise.} \end{aligned} \quad (4.8)$$

$$R_i(\boldsymbol{\theta}, \rho) = \sup \{R_i(\boldsymbol{\psi}) : \|\boldsymbol{\psi} - \boldsymbol{\theta}\| \leq \rho\} \quad (4.9)$$

$$\mathcal{V}_i(r) = \sup \{R_i(\boldsymbol{\theta}) : \|\boldsymbol{\theta}\| > r\}. \quad (4.10)$$

Following Hoadley (1971) we denote by $r_i(\boldsymbol{\theta})$, $r_i(\boldsymbol{\theta}, \rho)$ and $v_i(r)$ to be expectations of $R_i(\boldsymbol{\theta})$, $R_i(\boldsymbol{\theta}, \rho)$ and $\mathcal{V}_i(r)$ under $\boldsymbol{\theta}_0$; for any sequence $\{a_i; i = 1, 2, \dots\}$ we denote $\sum_{i=1}^n a_i/n$ by \bar{a}_n .

Hoadley (1971) proved that if the following regularity conditions are satisfied, then the MLE $\hat{\boldsymbol{\theta}}_n \xrightarrow{P} \boldsymbol{\theta}_0$:

- (1) $\boldsymbol{\Theta}$ is a closed subset of \mathbb{R}^{p+q+1} .
- (2) $f_i(X_i|\boldsymbol{\theta})$ is an upper semicontinuous function of $\boldsymbol{\theta}$, uniformly in i , a.s. $[P_{\boldsymbol{\theta}_0}]$.
- (3) There exist $\rho^* = \rho^*(\boldsymbol{\theta}) > 0$, $r > 0$ and $0 < K^* < \infty$ for which
 - (i) $E_{\boldsymbol{\theta}_0} [R_i(\boldsymbol{\theta}, \rho)]^2 \leq K^*$, $0 \leq \rho \leq \rho^*$;
 - (ii) $E_{\boldsymbol{\theta}_0} [\mathcal{V}_i(r)]^2 \leq K^*$.
- (4) (i) $\lim_{n \rightarrow \infty} \bar{r}_n(\boldsymbol{\theta}) < 0$, $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$;

$$(ii) \lim_{n \rightarrow \infty} \bar{v}_n(r) < 0.$$

(5) $R_i(\boldsymbol{\theta}, \rho)$ and $\mathcal{V}_i(r)$ are measurable functions of X_i .

Actually, conditions (3) and (4) can be weakened but these are more easily applicable (see Hoadley (1971) for details).

4.1.1 Verification of the regularity conditions

Since Θ is compact in our case, the first regularity condition clearly holds.

For the second regularity condition, note that given X_i , $f_i(X_i|\boldsymbol{\theta})$ is continuous by our assumptions (H1) – (H4), as already noted in Section 3.1.2; in fact, uniformly continuous in $\boldsymbol{\theta}$ in our case, since Θ is compact. Hence, for any given $\epsilon > 0$, there exists $\delta_i(\epsilon) > 0$, independent of $\boldsymbol{\theta}$, such that $\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\| < \delta_i(\epsilon)$ implies $|f(X_i|\boldsymbol{\theta}_1) - f(X_i|\boldsymbol{\theta}_2)| < \epsilon$. Now consider a strictly positive function $\delta_{x,T}(\epsilon)$, continuous in $x \in \mathfrak{X}$ and $T \in \mathfrak{T}$, such that $\delta_{x^i, T_i}(\epsilon) = \delta_i(\epsilon)$. Let $\delta(\epsilon) = \inf_{x \in \mathfrak{X}, T \in \mathfrak{T}} \delta_{x,T}(\epsilon)$. Since \mathfrak{X} and \mathfrak{T} are compact, it follows that $\delta(\epsilon) > 0$. Now it holds that $\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\| < \delta(\epsilon)$ implies $|f(X_i|\boldsymbol{\theta}_1) - f(X_i|\boldsymbol{\theta}_2)| < \epsilon$, for all i . Hence, the second regularity condition is satisfied.

Let us now focus attention on condition (3)(i).

$$\begin{aligned} R_i(\boldsymbol{\theta}) &= U_{i,\boldsymbol{\theta}} - \frac{V_{i,\boldsymbol{\theta}}}{2} - U_{i,\boldsymbol{\theta}_0} + \frac{V_{i,\boldsymbol{\theta}_0}}{2} \\ &\leq U_{i,\boldsymbol{\theta}} + \frac{V_{i,\boldsymbol{\theta}}}{2} - U_{i,\boldsymbol{\theta}_0} + \frac{V_{i,\boldsymbol{\theta}_0}}{2} \end{aligned} \quad (4.11)$$

Let us denote $\{\boldsymbol{\psi} \in \mathbb{R}^{p+q+1} : \|\boldsymbol{\psi} - \boldsymbol{\theta}\| \leq \rho\}$ by $S(\rho, \boldsymbol{\theta})$. Here $0 < \rho < \rho^*(\boldsymbol{\theta})$, and $\rho^*(\boldsymbol{\theta})$ is so small that $S(\rho, \boldsymbol{\theta}) \subset \Theta$ for all $\rho \in (0, \rho^*(\boldsymbol{\theta}))$. It then follows from (4.11) that

$$\sup_{\boldsymbol{\psi} \in S(\rho, \boldsymbol{\theta})} R_i(\boldsymbol{\psi}) \leq \sup_{\boldsymbol{\theta} \in S(\rho, \boldsymbol{\theta})} \left(U_{i,\boldsymbol{\theta}} + \frac{V_{i,\boldsymbol{\theta}}}{2} \right) - U_{i,\boldsymbol{\theta}_0} + \frac{V_{i,\boldsymbol{\theta}_0}}{2}. \quad (4.12)$$

The supremums in (4.12) are finite due to compactness of $S(\rho, \boldsymbol{\theta})$. Let the supremum be attained at some $\boldsymbol{\theta}^*$ where $\boldsymbol{\theta}^* = \boldsymbol{\theta}^*(X_i)$. Then, the expectation of the square of the upper bound can be calculated in the same way as (3.3) noting that $\bar{N}_{\boldsymbol{\theta}}$ in this case will be $S(\rho, \boldsymbol{\theta})$. Since under $P_{\boldsymbol{\theta}_0}$, finiteness of moments of all orders of each term in the upper bound is ensured by Lemma 10 of Maitra and Bhattacharya (2016a) it follows that

$$E_{\boldsymbol{\theta}_0} [R_i(\boldsymbol{\theta}, \rho)]^2 \leq K_i(\boldsymbol{\theta}), \quad (4.13)$$

where $K_i(\boldsymbol{\theta}) = K(x^i, T_i, \mathbf{z}_i, \boldsymbol{\theta})$, with $K(x, T, \mathbf{z}, \boldsymbol{\theta})$ being a continuous function of $(x, T, \mathbf{z}, \boldsymbol{\theta})$, continuity being again a consequence of Lemma 10 of Maitra and Bhattacharya (2016a). Since because of compactness of \mathfrak{X} , \mathfrak{T} and Θ ,

$$K_i(\boldsymbol{\theta}) \leq \sup_{x \in \mathfrak{X}, T \in \mathfrak{T}, \mathbf{z} \in \mathfrak{Z}, \boldsymbol{\theta} \in \Theta} K(x, T, \mathbf{z}, \boldsymbol{\theta}) < \infty,$$

regularity condition (3)(i) follows.

To verify condition (3)(ii), first note that we can choose $r > 0$ such that $\|\boldsymbol{\theta}_0\| < r$ and $\{\boldsymbol{\theta} \in \Theta : \|\boldsymbol{\theta}\| > r\} \neq \emptyset$. It then follows that $\sup_{\{\boldsymbol{\theta} \in \Theta : \|\boldsymbol{\theta}\| > r\}} R_i(\boldsymbol{\theta}) \leq \sup_{\boldsymbol{\theta} \in \Theta} R_i(\boldsymbol{\theta})$ for every $i \geq 1$. The right hand side is bounded by the same expression as the right hand side of (4.12), with only $S(\rho, \boldsymbol{\theta})$ replaced with Θ . The rest of the verification follows in the same way as verification of (3)(i).

To verify condition (4)(i) note that by (4.5)

$$\lim_{n \rightarrow \infty} \bar{r}_n = - \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \mathcal{K}_i(\boldsymbol{\theta}_0, \boldsymbol{\theta})}{n} = -\mathcal{K}(\boldsymbol{\theta}_0, \boldsymbol{\theta}) < 0 \quad \text{for } \boldsymbol{\theta} \neq \boldsymbol{\theta}_0. \quad (4.14)$$

In other words, (4)(i) is satisfied.

Verification of (4)(ii) follows exactly in a similar way as verified in Maitra and Bhattacharya (2016c) except that the concerned moment existence result follows from Lemma 10 of Maitra and Bhattacharya (2016a). Regularity condition (5) is seen to hold by the same arguments as in Maitra and Bhattacharya (2016c).

In other words, in the non-*iid* set-up in the non-*iid* SDE framework, the following theorem holds:

Theorem 6 Assume the non-*iid* SDE setup and conditions (H1) – (H4) and (H8). Then it holds that $\hat{\theta}_n \xrightarrow{P} \theta_0$.

4.2 Asymptotic normality of MLE in the non-*iid* set-up

Let $\zeta_i(x, \theta) = \log f_i(x|\theta)$; also, let $\zeta'_i(x, \theta)$ be the $(p+q+1) \times 1$ vector with k -th component $\zeta'_{i,k}(x, \theta) = \frac{\partial}{\partial \theta_k} \zeta_i(x, \theta)$, and let $\zeta''_i(x, \theta)$ be the $(p+q+1) \times (p+q+1)$ matrix with (k, l) -th element $\zeta''_{i,kl}(x, \theta) = \frac{\partial^2}{\partial \theta_k \partial \theta_l} \zeta_i(x, \theta)$.

For proving asymptotic normality in the non-*iid* framework, Hoadley (1971) assumed the following regularity conditions:

- (1) Θ is an open subset of \mathbb{R}^{p+q+1} .
- (2) $\hat{\theta}_n \xrightarrow{P} \theta_0$.
- (3) $\zeta'_i(X_i, \theta)$ and $\zeta''_i(X_i, \theta)$ exist a.s. $[P_{\theta_0}]$.
- (4) $\zeta''_i(X_i, \theta)$ is a continuous function of θ , uniformly in i , a.s. $[P_{\theta_0}]$, and is a measurable function of X_i .
- (5) $E_{\theta}[\zeta'_i(X_i, \theta)] = 0$ for $i = 1, 2, \dots$
- (6) $\mathcal{I}_i(\theta) = E_{\theta} [\zeta'_i(X_i, \theta) \zeta'_i(X_i, \theta)^T] = -E_{\theta} [\zeta''_i(X_i, \theta)]$, where for any vector y , y^T denotes the transpose of y .
- (7) $\bar{\mathcal{I}}_n(\theta) \rightarrow \bar{\mathcal{I}}(\theta)$ as $n \rightarrow \infty$ and $\bar{\mathcal{I}}(\theta)$ is positive definite.
- (8) $E_{\theta_0} \left| \zeta'_{i,k}(X_i, \theta_0) \right|^3 \leq K_2$, for some $0 < K_2 < \infty$.
- (9) There exist $\epsilon > 0$ and random variables $B_{i,kl}(X_i)$ such that
 - (i) $\sup \left\{ \left| \zeta''_{i,kl}(X_i, \psi) \right| : \|\psi - \theta_0\| \leq \epsilon \right\} \leq B_{i,kl}(X_i)$.
 - (ii) $E_{\theta_0} |B_{i,kl}(X_i)|^{1+\delta} \leq K_2$, for some $\delta > 0$.

Condition (8) can be weakened but is relatively easy to handle. Under the above regularity conditions, Hoadley (1971) prove that

$$\sqrt{n} (\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} N(\mathbf{0}, \bar{\mathcal{I}}^{-1}(\theta_0)). \quad (4.15)$$

4.2.1 Validation of asymptotic normality of MLE in the non-*iid* SDE set-up

Condition (1) holds also for compact Θ ; see Maitra and Bhattacharya (2016c). Condition (2) is a simple consequence of Theorem 6.

Conditions (3), (5) and (6) are clearly valid in our case because of interchangability of differentiation and integration, which follows due to (H1) – (H4), (H7) and Theorem B.4 of Rao (2013). Condition (4) can be verified in exactly the same way as condition (2) of Section 4.1 is verified; measurability of $\zeta''_i(X_i, \theta)$ follows due its continuity with respect to X_i . Condition (7) simply follows from (4.7). Compactness, continuity, and finiteness of moments guaranteed by Lemma 10 of Maitra and Bhattacharya (2016a) imply conditions (8), (9)(i) and 9(ii).

In other words, in our non-*iid* SDE case we have the following theorem on asymptotic normality.

Theorem 7 Assume the non-iid SDE setup and conditions (H1) – (H8). Then (4.15) holds.

5 Consistency and asymptotic normality of the Bayesian posterior in the iid set-up

5.1 Consistency of the Bayesian posterior distribution

As in Maitra and Bhattacharya (2015) here we exploit Theorem 7.80 presented in Schervish (1995), stated below, to show posterior consistency.

Theorem 8 (Schervish (1995)) Let $\{X_n\}_{n=1}^\infty$ be conditionally iid given θ with density $f_1(x|\theta)$ with respect to a measure ν on a space $(\mathcal{X}^1, \mathcal{B}^1)$. Fix $\theta_0 \in \Theta$, and define, for each $\mathbf{M} \subseteq \Theta$ and $x \in \mathcal{X}^1$,

$$Z(\mathbf{M}, x) = \inf_{\psi \in \mathbf{M}} \log \frac{f_1(x|\theta_0)}{f_1(x|\psi)}.$$

Assume that for each $\theta \neq \theta_0$, there is an open set \mathbf{N}_θ such that $\theta \in \mathbf{N}_\theta$ and that $E_{\theta_0} Z(\mathbf{N}_\theta, X_i) > -\infty$. Also assume that $f_1(x|\cdot)$ is continuous at θ for every θ , a.s. $[P_{\theta_0}]$. For $\epsilon > 0$, define $\mathbf{C}_\epsilon = \{\theta : \mathcal{K}_1(\theta_0, \theta) < \epsilon\}$, where

$$\mathcal{K}_1(\theta_0, \theta) = E_{\theta_0} \left(\log \frac{f_1(X_1|\theta_0)}{f_1(X_1|\theta)} \right) \quad (5.1)$$

is the Kullback-Leibler divergence measure associated with observation X_1 . Let π be a prior distribution such that $\pi(\mathbf{C}_\epsilon) > 0$, for every $\epsilon > 0$. Then, for every $\epsilon > 0$ and open set \mathcal{N}_0 containing \mathbf{C}_ϵ , the posterior satisfies

$$\lim_{n \rightarrow \infty} \pi_n(\mathcal{N}_0 | \mathbf{X}_1, \dots, \mathbf{X}_n) = 1, \quad \text{a.s. } [P_{\theta_0}]. \quad (5.2)$$

5.1.1 Verification of posterior consistency

The condition $E_{\theta_0} Z(\mathbf{N}_\theta, X_i) > -\infty$ of the above theorem is verified in the context of Theorem 1 in Section 3.1.2. Continuity of the Kullback-Liebler divergence follows easily from Lemma 10 of Maitra and Bhattacharya (2016a). The rest of the verification is the same as that of Maitra and Bhattacharya (2015).

Hence, (5.2) holds in our case with any prior with positive, continuous density with respect to the Lebesgue measure. We summarize this result in the form of a theorem, stated below.

Theorem 9 Assume the iid set-up and conditions (H1) – (H4). Let the prior distribution π of the parameter θ satisfy $\frac{d\pi}{d\nu} = g$ almost everywhere on Θ , where $g(\theta)$ is any positive, continuous density on Θ with respect to the Lebesgue measure ν . Then the posterior (1.2) is consistent in the sense that for every $\epsilon > 0$ and open set \mathcal{N}_0 containing \mathbf{C}_ϵ , the posterior satisfies

$$\lim_{n \rightarrow \infty} \pi_n(\mathcal{N}_0 | \mathbf{X}_1, \dots, \mathbf{X}_n) = 1, \quad \text{a.s. } [P_{\theta_0}]. \quad (5.3)$$

5.2 Asymptotic normality of the Bayesian posterior distribution

As in Maitra and Bhattacharya (2015), we make use of Theorem 7.102 in conjunction with Theorem 7.89 provided in Schervish (1995). These theorems make use of seven regularity conditions, of which only the first four, stated below, will be required for the iid set-up.

5.2.1 Regularity conditions – iid case

- (1) The parameter space is $\Theta \subseteq \mathbb{R}^{q+1}$.
- (2) θ_0 is a point interior to Θ .

- (3) The prior distribution of θ has a density with respect to Lebesgue measure that is positive and continuous at θ_0 .
- (4) There exists a neighborhood $\mathcal{N}_0 \subseteq \Theta$ of θ_0 on which $\ell_n(\theta) = \log f(X_1, \dots, X_n|\theta)$ is twice continuously differentiable with respect to all co-ordinates of θ , a.s. $[P_{\theta_0}]$.

Before proceeding to justify asymptotic normality of our posterior, we furnish the relevant theorem below (Theorem 7.102 of Schervish (1995)).

Theorem 10 (Schervish (1995)) *Let $\{X_n\}_{n=1}^\infty$ be conditionally iid given θ . Assume the above four regularity conditions; also assume that there exists $H_r(x, \theta)$ such that, for each $\theta_0 \in \text{int}(\Theta)$ and each k, j ,*

$$\sup_{\|\theta - \theta_0\| \leq r} \left| \frac{\partial^2}{\partial \theta_k \partial \theta_j} \log f_1(x|\theta_0) - \frac{\partial^2}{\partial \theta_k \partial \theta_j} \log f_1(x|\theta) \right| \leq H_r(x, \theta_0), \quad (5.4)$$

with

$$\lim_{r \rightarrow 0} E_{\theta_0} H_r(X, \theta_0) = 0. \quad (5.5)$$

Further suppose that the conditions of Theorem 8 hold, and that the Fisher's information matrix $\mathcal{I}(\theta_0)$ is positive definite. Now denoting by $\hat{\theta}_n$ the MLE associated with n observations, let

$$\Sigma_n^{-1} = \begin{cases} -\ell_n''(\hat{\theta}_n) & \text{if the inverse and } \hat{\theta}_n \text{ exist} \\ \mathbb{I}_{q+1} & \text{if not,} \end{cases} \quad (5.6)$$

where for any t ,

$$\ell_n''(t) = \left(\left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \ell_n(\theta) \Big|_{\theta=t} \right) \right), \quad (5.7)$$

and \mathbb{I}_{q+1} is the identity matrix of order $q + 1$. Thus, Σ_n^{-1} is the observed Fisher's information matrix.

Letting $\Psi_n = \Sigma_n^{-1/2} (\theta - \hat{\theta}_n)$, it follows that for each compact subset B of \mathbb{R}^{q+1} and each $\epsilon > 0$, it holds that

$$\lim_{n \rightarrow \infty} P_{\theta_0} \left(\sup_{\Psi_n \in B} \left| \pi_n(\Psi_n | X_1, \dots, X_n) - \tilde{\phi}(\Psi_n) \right| > \epsilon \right) = 0, \quad (5.8)$$

where $\tilde{\phi}(\cdot)$ denotes the density of the standard normal distribution.

5.2.2 Verification of posterior normality

Observe that the four regularity conditions of Section 5.2.1 trivially hold. The remaining conditions of Theorem 10 are verified in the context of Theorem 3 in Section 3.2.2. We summarize this result in the form of the following theorem.

Theorem 11 *Assume the iid set-up and conditions (H1) – (H7). Let the prior distribution π of the parameter θ satisfy $\frac{d\pi}{d\nu} = g$ almost everywhere on Θ , where $g(\theta)$ is any density with respect to the Lebesgue measure ν which is positive and continuous at θ_0 . Then, letting $\Psi_n = \Sigma_n^{-1/2} (\theta - \hat{\theta}_n)$, it follows that for each compact subset B of \mathbb{R}^{q+1} and each $\epsilon > 0$, it holds that*

$$\lim_{n \rightarrow \infty} P_{\theta_0} \left(\sup_{\Psi_n \in B} \left| \pi_n(\Psi_n | X_1, \dots, X_n) - \tilde{\phi}(\Psi_n) \right| > \epsilon \right) = 0. \quad (5.9)$$

6 Consistency and asymptotic normality of the Bayesian posterior in the non-*iid* set-up

For consistency and asymptotic normality in the non-*iid* Bayesian framework we utilize the result presented in Choi and Schervish (2007) and Theorem 7.89 of Schervish (1995), respectively.

6.1 Posterior consistency in the non-*iid* set-up

We consider the following extra assumption for our purpose.

(H9) There exist strictly positive functions $\alpha_1^*(x, T, \mathbf{z}, \boldsymbol{\theta})$ and $\alpha_2^*(x, T, \mathbf{z}, \boldsymbol{\theta})$ continuous in $(x, T, \mathbf{z}, \boldsymbol{\theta})$, such that for any $(x, T, \mathbf{z}, \boldsymbol{\theta})$,

$$E_{\boldsymbol{\theta}} [\exp \{ \alpha_1^*(x, T, \mathbf{z}, \boldsymbol{\theta}) U_{\boldsymbol{\theta}}(x, T, \mathbf{z}) \}] < \infty,$$

and

$$E_{\boldsymbol{\theta}} [\exp \{ \alpha_2^*(x, T, \mathbf{z}, \boldsymbol{\theta}) V_{\boldsymbol{\theta}}(x, T, \mathbf{z}) \}] < \infty,$$

Now, let

$$\alpha_{1,\min}^* = \inf_{x \in \mathfrak{X}, T \in \mathfrak{T}, \mathbf{z} \in \mathfrak{Z}, \boldsymbol{\theta} \in \boldsymbol{\Theta}} \alpha_1^*(x, T, \mathbf{z}, \boldsymbol{\theta}) \quad (6.1)$$

$$\alpha_{2,\min}^* = \inf_{x \in \mathfrak{X}, T \in \mathfrak{T}, \mathbf{z} \in \mathfrak{Z}, \boldsymbol{\theta} \in \boldsymbol{\Theta}} \alpha_2^*(x, T, \mathbf{z}, \boldsymbol{\theta}) \quad (6.2)$$

and

$$\alpha = \min \{ \alpha_{1,\min}^*, \alpha_{2,\min}^*, c^* \}, \quad (6.3)$$

where $0 < c^* < 1/16$.

Compactness ensures that $\alpha_{1,\min}^*, \alpha_{2,\min}^* > 0$, so that $0 < \alpha < 1/16$. It also holds due to compactness that for $\boldsymbol{\theta} \in \boldsymbol{\Theta}$,

$$\sup_{x \in \mathfrak{X}, T \in \mathfrak{T}, \mathbf{z} \in \mathfrak{Z}, \boldsymbol{\theta} \in \boldsymbol{\Theta}} E_{\boldsymbol{\theta}} [\exp \{ \alpha U_{\boldsymbol{\theta}}(x, T, \mathbf{z}) \}] < \infty. \quad (6.4)$$

and

$$\sup_{x \in \mathfrak{X}, T \in \mathfrak{T}, \mathbf{z} \in \mathfrak{Z}, \boldsymbol{\theta} \in \boldsymbol{\Theta}} E_{\boldsymbol{\theta}} [\exp \{ \alpha V_{\boldsymbol{\theta}}(x, T, \mathbf{z}) \}] < \infty. \quad (6.5)$$

This choice of α ensuring (6.4) and (6.5) will be useful in verification of the conditions of Theorem 12, which we next state.

Theorem 12 (Choi and Schervish (2007)) *Let $\{X_i\}_{i=1}^{\infty}$ be independently distributed with densities $\{f_i(\cdot|\boldsymbol{\theta})\}_{i=1}^{\infty}$, with respect to a common σ -finite measure, where $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, a measurable space. The densities $f_i(\cdot|\boldsymbol{\theta})$ are assumed to be jointly measurable. Let $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$ and let $P_{\boldsymbol{\theta}_0}$ be the joint distribution of $\{X_i\}_{i=1}^{\infty}$ when $\boldsymbol{\theta}_0$ is the true value of $\boldsymbol{\theta}$. Let $\{\boldsymbol{\Theta}_n\}_{n=1}^{\infty}$ be a sequence of subsets of $\boldsymbol{\Theta}$. Let $\boldsymbol{\theta}$ have prior π on $\boldsymbol{\Theta}$. Define the following:*

$$\begin{aligned} \Lambda_i(\boldsymbol{\theta}_0, \boldsymbol{\theta}) &= \log \frac{f_i(X_i|\boldsymbol{\theta}_0)}{f_i(X_i|\boldsymbol{\theta})}, \\ \mathcal{K}_i(\boldsymbol{\theta}_0, \boldsymbol{\theta}) &= E_{\boldsymbol{\theta}_0} (\Lambda_i(\boldsymbol{\theta}_0, \boldsymbol{\theta})) \\ \varrho_i(\boldsymbol{\theta}_0, \boldsymbol{\theta}) &= \text{Var}_{\boldsymbol{\theta}_0} (\Lambda_i(\boldsymbol{\theta}_0, \boldsymbol{\theta})). \end{aligned}$$

Make the following assumptions:

(1) Suppose that there exists a set \mathbf{B} with $\pi(\mathbf{B}) > 0$ such that

$$(i) \sum_{i=1}^{\infty} \frac{\varrho_i(\boldsymbol{\theta}_0, \boldsymbol{\theta})}{i^2} < \infty, \quad \forall \boldsymbol{\theta} \in \mathbf{B},$$

(ii) For all $\epsilon > 0$, $\pi(\mathbf{B} \cap \{\boldsymbol{\theta} : \mathcal{K}_i(\boldsymbol{\theta}_0, \boldsymbol{\theta}) < \epsilon, \forall i\}) > 0$.

(2) Suppose that there exist test functions $\{\Phi_n\}_{n=1}^\infty$, sets $\{\Omega_n\}_{n=1}^\infty$ and constants $C_1, C_2, c_1, c_2 > 0$ such that

- (i) $\sum_{n=1}^\infty E_{\theta_0} \Phi_n < \infty$,
- (ii) $\sup_{\theta \in \Theta_n^c \cap \Omega_n} E_\theta (1 - \Phi_n) \leq C_1 e^{-c_1 n}$,
- (iii) $\pi(\Omega_n^c) \leq C_2 e^{-c_2 n}$.

Then,

$$\pi_n(\theta \in \Theta_n^c | X_1, \dots, X_n) \rightarrow 0 \quad a.s. [P_{\theta_0}]. \quad (6.6)$$

6.1.1 Validation of posterior consistency

Firstly, note that, $f_i(X_i|\theta)$ is given by (2.4). From the proof of Theorem 6 it follows that $\left| \log \frac{f_i(X_i|\theta_0)}{f_i(X_i|\theta)} \right|$ has an upper bound which has finite expectation and square of expectation under θ_0 , and is uniform for all $\theta \in B$, where B is any compact subset of Θ . Hence, for each i , $\varrho_i(\theta_0, \theta)$ is finite. Using compactness, Lemma 10 of Maitra and Bhattacharya (2016a) and arguments similar to that of Section 3.1.1 of Maitra and Bhattacharya (2015), it easily follows that $\varrho_i(\theta_0, \theta) < \kappa$, for some $0 < \kappa < \infty$, uniformly in i . Hence, choosing a prior that gives positive probability to the set B , it follows that for all $\theta \in B$,

$$\sum_{i=1}^\infty \frac{\varrho_i(\theta_0, \theta)}{i^2} < \kappa \sum_{i=1}^\infty \frac{1}{i^2} < \infty.$$

Hence, condition (1)(i) holds. Also note that (1)(ii) can be verified similarly as the verification of Theorem 5 of Maitra and Bhattacharya (2015).

We now verify conditions (2)(i), (2)(ii) and (2)(iii). We let $\Omega_n = (\Omega_{1n} \times \mathbb{R}^{p+1})$, where $\Omega_{1n} = \{\beta : \|\beta\| < M_n\}$, where $M_n = O(e^n)$. Note that

$$\pi(\Omega_n^c) = \pi(\Omega_{1n}^c) = \pi(\|\beta\| \geq M_n) < E_\pi(\|\beta\|) M_n^{-1}, \quad (6.7)$$

so that (2)(iii) holds, assuming that the prior π is such that the expectation $E_\pi(\|\beta\|)$ is finite.

The verification of 2(i) can be checked in as in Maitra and Bhattacharya (2015) except the relevant changes. So, here we only mention the corresponding changes, skipping detailed verification.

Kolmogorov's strong law of large numbers for the non-*iid* case (see, for example, Serfling (1980)), holds in our problem due to finiteness of the moments of $U_\theta(x, T, z)$ and $V_\theta(x, T, z)$ for every x, T, z and θ belonging to the respective compact spaces. Moreover, existence and boundedness of the third order derivative of $\ell_n(\theta)$ with respect to its components is ensured by assumption (H7) along with compactness assumptions. The results stated in Maitra and Bhattacharya (2016a) concerned with continuity and finiteness of the moments of $U_\theta(x, T, z)$ and $V_\theta(x, T, z)$ for every x, T, z and θ belonging to their respective compact spaces are needed here. The lower bound of $\log f_i(X_i|\theta_0) - \log f_i(X_i|\hat{\theta}_n)$ is denoted by $C_3(U_i, V_i, \hat{\theta}_n)$ where

$$C_3(U_i, V_i, \hat{\theta}_n) = U_{i, \theta_0} - \frac{V_{i, \theta_0}}{2} - U_{i, \hat{\theta}_n} - \frac{V_{i, \hat{\theta}_n}}{2}$$

The rest of the verification is same as that of Maitra and Bhattacharya (2015) along with assumption (H9).

To verify condition 2(ii) we define $\Theta_n = \Theta_\delta = \{(\beta, \xi) : \mathcal{K}(\theta, \theta_0) < \delta\}$, where $\mathcal{K}(\theta, \theta_0)$, defined as in (4.6), is the proper Kullback-Leibler divergence. This verification is again similar to that of Maitra and Bhattacharya (2015). The result can be summarized in the form of the following theorem.

Theorem 13 Assume the non-*iid* SDE set-up. Also assume conditions (H1) – (H9). For any $\delta > 0$, let $\Theta_\delta = \{(\beta, \xi) : \mathcal{K}(\theta, \theta_0) < \delta\}$, where $\mathcal{K}(\theta, \theta_0)$, defined as in (4.6), is the proper Kullback-Leibler

divergence. Let the prior distribution π of the parameter θ satisfy $\frac{d\pi}{d\nu} = h$ almost everywhere on Θ , where $h(\theta)$ is any positive, continuous density on Θ with respect to the Lebesgue measure ν . Then,

$$\pi_n(\theta \in \Theta_\delta^c | X_1, \dots, X_n) \rightarrow 0 \quad a.s. [P_{\theta_0}]. \quad (6.8)$$

6.2 Asymptotic normality of the posterior distribution in the non-*iid* set-up

Below we present the three regularity conditions that are needed in the non-*iid* set-up in addition to the three conditions already stated in Section 5.2.1, for asymptotic normality given by (5.8).

6.2.1 Extra regularity conditions in the non-*iid* set-up

- (5) The largest eigenvalue of Σ_n goes to zero in probability.
- (6) For $\delta > 0$, define $\mathcal{N}_0(\delta)$ to be the open ball of radius δ around θ_0 . Let ρ_n be the smallest eigenvalue of Σ_n . If $\mathcal{N}_0(\delta) \subseteq \Theta$, there exists $K(\delta) > 0$ such that

$$\lim_{n \rightarrow \infty} P_{\theta_0} \left(\sup_{\theta \in \Theta \setminus \mathcal{N}_0(\delta)} \rho_n [\ell_n(\theta) - \ell_n(\theta_0)] < -K(\delta) \right) = 1. \quad (6.9)$$

- (7) For each $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\lim_{n \rightarrow \infty} P_{\theta_0} \left(\sup_{\theta \in \mathcal{N}_0(\delta(\epsilon)), \|\gamma\|=1} \left| 1 + \gamma^T \Sigma_n^{\frac{1}{2}} \ell_n''(\theta) \Sigma_n^{\frac{1}{2}} \gamma \right| < \epsilon \right) = 1. \quad (6.10)$$

6.2.2 Verification of the regularity conditions

Assumptions (H1) – (H9), along with Kolmogorov's strong law of large numbers, are sufficient for the regularity conditions to hold; the arguments remain similar as those in Section 3.2.2 of Maitra and Bhattacharya (2015). We provide our result in the form of the following theorem.

Theorem 14 Assume the non-*iid* set-up and conditions (H1) – (H9). Let the prior distribution π of the parameter θ satisfy $\frac{d\pi}{d\nu} = h$ almost everywhere on Θ , where $h(\theta)$ is any density with respect to the Lebesgue measure ν which is positive and continuous at θ_0 . Then, letting $\Psi_n = \Sigma_n^{-1/2} (\theta - \hat{\theta}_n)$, for each compact subset B of \mathbb{R}^{p+q+1} and each $\epsilon > 0$, the following holds:

$$\lim_{n \rightarrow \infty} P_{\theta_0} \left(\sup_{\Psi_n \in B} \left| \pi_n(\Psi_n | X_1, \dots, X_n) - \tilde{\phi}(\Psi_n) \right| > \epsilon \right) = 0. \quad (6.11)$$

7 Random effects SDE model

We now consider the following system of SDE models for $i = 1, 2, \dots, n$:

$$dX_i(t) = \phi_{\xi^i}(t) b_{\beta}(X_i(t)) dt + \sigma(X_i(t)) dW_i(t) \quad (7.1)$$

Note that this model is the same as described in Section 2 except that the parameters ξ^i now depend upon i . Indeed, now $\phi_{\xi^i}(t)$ is given by

$$\phi_{\xi^i}(t) = \phi_{\xi^i}(z_i(t)) = \xi_0^i + \xi_1^i g_1(z_{i1}(t)) + \xi_2^i g_2(z_{i2}(t)) + \dots + \xi_p^i g_p(z_{ip}(t)), \quad (7.2)$$

where $\xi^i = (\xi_0^i, \xi_1^i, \dots, \xi_p^i)^T$ is the random effect corresponding to the i -th individual for $i = 1, \dots, n$, and $z_i(t)$ is the same as in Section 2.1. We let $b_{\beta}(X_i(t), \phi_{\xi^i}) = \phi_{\xi^i}(t) b_{\beta}(X_i(t))$. Note that our

likelihood is the product over $i = 1, \dots, n$, of the following individual densities:

$$f_{i,\xi^i,\beta}(X_i) = \exp\left(U_{i,\xi^i,\beta} - \frac{V_{i,\xi^i,\beta}}{2}\right),$$

where

$$U_{i,\xi^i,\beta} = \int_0^{T_i} \frac{\phi_{\xi^i}(s)b_{\beta}(X_i(s))}{\sigma^2(X_i(s))} dX_i(s) \quad \text{and} \quad V_{i,\xi^i,\beta} = \int_0^{T_i} \frac{\phi_{\xi^i}^2(s)b_{\beta}^2(X_i(s))}{\sigma^2(X_i(s))} ds.$$

Now, let $m^{\beta}(z(t), x(t)) = (m_0^{\beta}, m_1^{\beta}(z_1(t), x(t)), \dots, m_p^{\beta}(z_p(t), x(t)))^T$ be a function from $\mathfrak{Z} \times \mathbb{R} \rightarrow \mathbb{R}^{p+1}$ where $m_0^{\beta} \equiv 1$ and $m_k^{\beta}(z(t), x(t)) = g_k(z_k(t))b_{\beta}(x(t))$; $k = 1, \dots, p$. With this notation, the likelihood can be re-written as the product over $i = 1, \dots, n$, of the following:

$$f_{i,\xi^i,\beta}(X_i) = \exp((\xi^i)^T A_i^{\beta} - (\xi^i)^T B_i^{\beta} \xi^i) \quad (7.3)$$

where

$$A_i^{\beta} = \int_0^{T_i} \frac{m^{\beta}(z(s), X_i(s))}{\sigma^2(X_i(s))} dX_i(s) \quad (7.4)$$

and

$$B_i^{\beta} = \int_0^{T_i} \frac{m^{\beta}(z(s), X_i(s)) (m^{\beta})^T(z(s), X_i(s))}{\sigma^2(X_i(s))} ds \quad (7.5)$$

are $(p+1) \times 1$ random vectors and positive definite $(p+1) \times (p+1)$ random matrices respectively.

We assume that ξ^i are *iid* Gaussian vectors, with expectation vector μ and covariance matrix $\Sigma \in \mathcal{S}_{p+1}(\mathbb{R})$ where $\mathcal{S}_{p+1}(\mathbb{R})$ is the set of real positive definite symmetric matrices of order $p+1$. The parameter set is denoted by $\theta = (\mu, \Sigma, \beta) \in \Theta \subset \mathbb{R}^{p+1} \times \mathcal{S}_{p+1}(\mathbb{R}) \times \mathbb{R}^q$.

To obtain the likelihood involving θ we refer to the multidimensional random effects set-up of Delattre *et al.* (2013). Following Lemma 2 of Delattre *et al.* (2013) it then follows in our case that, for each $i \geq 1$ and for all θ , $B_i^{\beta} + \Sigma^{-1}$, $\mathbb{I}_{p+1} + B_i^{\beta} \Sigma$, $\mathbb{I}_{p+1} + \Sigma B_i^{\beta}$ are invertible.

Setting $(R_i^{\beta})^{-1} = (\mathbb{I}_{p+1} + B_i^{\beta} \Sigma)^{-1} B_i^{\beta}$ we obtain

$$\begin{aligned} f_i(X_i|\theta) &= \frac{1}{\sqrt{\det(\mathbb{I}_{p+1} + B_i^{\beta} \Sigma)}} \exp\left(-\frac{1}{2} \left(\mu - (B_i^{\beta})^{-1} A_i^{\beta}\right)^T (R_i^{\beta})^{-1} \left(\mu - (B_i^{\beta})^{-1} A_i^{\beta}\right)\right) \\ &\quad \times \exp\left(\frac{1}{2} (A_i^{\beta})^T (B_i^{\beta})^{-1} A_i^{\beta}\right) \end{aligned} \quad (7.6)$$

as our desired likelihood after integrating (7.3) with respect to the distribution of ξ^i .

With reference to Delattre *et al.* (2013) in our case

$$\gamma_i(\theta) = (\mathbb{I}_{p+1} + \Sigma B_i^{\beta})^{-1} (A_i^{\beta} - B_i^{\beta} \mu) \quad \text{and} \quad I_i(\Sigma) = (\mathbb{I}_{p+1} + \Sigma B_i^{\beta})^{-1} B_i^{\beta}.$$

Hence, Proposition (10)(i) of Delattre *et al.* (2013) can be seen to hold here in a similar way by replacing U_i and V_i by A_i^{β} and B_i^{β} respectively.

Asymptotic investigation regarding consistency and asymptotic normality of *MLE* and Bayesian posterior consistency and asymptotic posterior normality in both *iid* and non-*iid* set-ups can be established as in the one dimensional cases in Maitra and Bhattacharya (2016c) and Maitra and Bhattacharya (2015) with proper multivariate modifications by replacing U_i and V_i with A_i^{β} and B_i^{β} respectively, and exploiting assumptions (H1) – (H9).

8 Summary and conclusion

In *SDE* based random effects model framework, Delattre *et al.* (2013) considered the linearity assumption in the drift function given by $b(x, \phi_i) = \phi_i b(x)$, assuming ϕ_i to be Gaussian random variables with mean μ and variance ω^2 , and obtained a closed form expression of the likelihood of the above parameters. Assuming the *iid* set-up, they proved convergence in probability and asymptotic normality of the maximum likelihood estimator of the parameters.

Maitra and Bhattacharya (2016a) and Maitra and Bhattacharya (2016b) extended their model by incorporating time-varying covariates in ϕ_i and allowing $b(x)$ to depend upon unknown parameters, but rather than inference regarding the parameters, they developed asymptotic model selection theory based on Bayes factors for their purposes. In this paper, we developed asymptotic theories for parametric inference for both classical and Bayesian paradigms under the fixed effects set-up, and provided relevant discussion of asymptotic inference on the parameters in the random effects set-up.

As our previous investigations (Maitra and Bhattacharya (2016c), Maitra and Bhattacharya (2015), for instance), in this work as well we distinguished the non-*iid* set-up from the *iid* case, the latter corresponding to the system of *SDE*'s with same initial values, time to main, but with no covariates. However, as already noted, this still provides a generalization to the *iid* set-up of Delattre *et al.* (2013) through generalization of $b(x)$ to $b_\beta(x)$; β being a set of unknown parameters. Under suitable assumptions we obtained strong consistency and asymptotic normality of the *MLE* under the *iid* set-up and weak consistency and asymptotic normality under the non-*iid* situation. Besides, we extended our classical asymptotic theory to the Bayesian framework, for both *iid* and non-*iid* situations. Specifically, we proved posterior consistency and asymptotic posterior normality, for both *iid* and non-*iid* set-ups.

In our knowledge, ours is the first-time effort regarding asymptotic inference, either classical or Bayesian, in systems of *SDE*'s under the presence of time-varying covariates.

Acknowledgment

The first author gratefully acknowledges her CSIR Fellowship, Govt. of India.

References

- Choi, T. and Schervish, M. J. (2007). On Posterior Consistency in Nonparametric Regression Problems. *Journal of Multivariate Analysis*, **98**, 1969–1987.
- Delattre, M., Genon-Catalot, V., and Samson, A. (2013). Maximum Likelihood Estimation for Stochastic Differential Equations with Random Effects. *Scandinavian Journal of Statistics*, **40**, 322–343.
- Hoadley, B. (1971). Asymptotic Properties of Maximum Likelihood Estimators for the Independent not Identically Distributed Case. *The Annals of Mathematical Statistics*, **42**, 1977–1991.
- Leander, J., Almquist, J., Ahlström, C., Gabrielsson, J., and Jirstrand, M. (2015). Mixed Effects Modeling Using Stochastic Differential Equations: Illustrated by Pharmacokinetic Data of Nicotinic Acid in Obese Zucker Rats. *The AAPS Journal*, **17**, 586–596.
- Maitra, T. and Bhattacharya, S. (2015). On Bayesian Asymptotics in Stochastic Differential Equations with Random Effects. *Statistics and Probability Letters*, **103**, 148–159. Also available at “<http://arxiv.org/abs/1407.3971>”.
- Maitra, T. and Bhattacharya, S. (2016a). Asymptotic Theory of Bayes Factor in Stochastic Differential Equations: Part I. Submitted.
- Maitra, T. and Bhattacharya, S. (2016b). Asymptotic Theory of Bayes Factor in Stochastic Differential Equations: Part II. Submitted.

- Maitra, T. and Bhattacharya, S. (2016c). On Asymptotics Related to Classical Inference in Stochastic Differential Equations with Random Effects. *Statistics and Probability Letters*, **110**, 278–288. Also available at “<http://arxiv.org/abs/1407.3968>”.
- Oravecz, Z., Tuerlinckx, F., and Vandekerckhove, J. (2011). A hierarchical latent stochastic differential equation model for affective dynamics. *Psychological Methods*, **16**, 468–490.
- Overgaard, R. V., Jonsson, N., Tornøe, C. W., and Madsen, H. (2005). Non-Linear Mixed-Effects Models with Stochastic Differential Equations: Implementation of an Estimation Algorithm. *Journal of Pharmacokinetics and Pharmacodynamics*, **32**, 85–107.
- Rao, B. (2013). *Semimartingales and their Statistical Inference*. Chapman and Hall/CRC, Boca Ratan.
- Schervish, M. J. (1995). *Theory of Statistics*. Springer-Verlag, New York.
- Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*. John Wiley & Sons, Inc., New York.